Complex Numbers

Throughout the following, assume that $z = a + bi = re^{i\phi}$ is an arbitrary complex number with a, b, r, and ϕ real ($r \geq 0$). Also assume that c is an arbitrary real number. These same assumptions apply if symbols are subscripted (e.g., $z_1 = a_1 + b_1 i$, etc).

Real and imaginary parts

$$z = a + bi \qquad a = \Re(z) \qquad b = \Im(z) \qquad |z| = \sqrt{a^2 + b^2}$$

Addition, subtraction, and multiplication (division later)

$$z_1 \pm z_2 = (a_1 \pm a_2) + (b_1 \pm b_2)i \qquad z_1 z_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i$$

$$i^2 = -1 \qquad iz = b - ai \qquad cz = (ca) + (cb)i$$

Complex conjugate

$$z^* = a - bi \qquad \Re(z^*) = \Re(z) \qquad \Im(z^*) = -\Im(z) \qquad c^* = c \qquad i^* = -i \qquad (z^*)^* = z$$

$$(z_1 \pm z_2)^* = z_1^* \pm z_2^* \qquad (z_1 z_2)^* = z_1^* z_2^* \qquad (z_1/z_2)^* = z_1^*/z_2^*$$

$$\Re(z) = (z + z^*)/2 \qquad \Im(z) = (z - z^*)/(2i) \qquad |z|^2 = z^* z$$

Division

$$z^{-1} = 1/z = z^*/|z|^2 \qquad 1/i = -i$$

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{|z_2|^2} = \left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} \right) + \left(\frac{b_1 a_2 - a_1 b_2}{a_2^2 + b_2^2} \right) i$$

Magnitudes

$$|z_1 z_2| = |z_1||z_2| \qquad |z_1/z_2| = |z_1|/|z_2| \qquad |z^c| = |z|^c \qquad |z^*| = |z|$$

$$||z_1| - |z_2|| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$$

Euler's identity

$$e^{i\phi} = \cos(\phi) + i\sin(\phi) \qquad e^{-i\phi} = \cos(\phi) - i\sin(\phi) = (e^{i\phi})^*$$

$$\cos(\phi) = (e^{i\phi} + e^{-i\phi})/2 \qquad \sin(\phi) = (e^{i\phi} - e^{-i\phi})/(2i) \qquad |e^{i\phi}| = 1$$

Polar representation

$$z = a + bi = re^{i\phi} \qquad a = r\cos(\phi) \qquad b = r\sin(\phi) \qquad r = \sqrt{a^2 + b^2} = |z| \qquad \tan(\phi) = b/a$$

$$z_1 z_2 = r_1 r_2 e^{i(\phi_1 + \phi_2)} \qquad z_1/z_2 = (r_1/r_2)e^{i(\phi_1 - \phi_2)} \qquad z^c = r^c e^{i(c\phi)} \qquad z^* = re^{-i\phi}$$